Simple Probability

Probability predicts how likely an event is. "Simple" probability predicts the chances of a single event. Probabilities are often given in percent format, but they can also be in a fraction, decimal, or ratio form. If something has a 100% probability, it will certainly happen. If something has a 0% probability, it will definitely not happen.

The formula for simple probability is: $\frac{\#\ of\ successful\ outcomes}{total\ \#\ of\ possible\ outcomes}$. A "successful" outcome occurs when the result matches the desired situation. For example, suppose we are going to flip a coin one time. We want to know the probability that the coin will land on heads. There are a total of two possible outcomes: heads and tails. The successful outcome, the one we want, is heads, is just one outcome. So the probability of a coin toss landing on heads is:

Probability of Heads = $\frac{\#\ of\ successful\ outcomes}{total\ \#\ of\ possible\ outcomes} = \frac{1}{2}$ = 50%

We say it's a "1 in 2 chance" or "50% chance."

For another example, suppose we choose one card from a deck, and we want to know the probability that it will be a diamond. There are 52 total possible outcomes, because the deck has 52 cards. There are 13 successful outcomes, because there are 13 diamond cards. The probability is:

Probability of Diamonds = $\frac{\#\ of\ successful\ outcomes}{total\ \#\ of\ possible\ outcomes} = \frac{13}{52}$ = 25%

There is a difference between simple "theoretical" probability like this and "experimental" or "empirical" probability. In theoretical probability, we predict what should happen. In experimental probability, we actually perform the event and record what happens. Suppose we toss a coin ten times. Theoretically, it should land on heads exactly five times and on tails exactly five times. But in reality, the coin might land on heads six times and on tails four times. The more times we toss the coin, the closer the experimental probability will get to the theoretical results.

Compound Probability

"Compound" probability is the probability of two or more events. Adding additional events changes the way we measure probability in two ways.

The first change with multiple events is the concept of dependency. Independent events do not affect each other. Examples of independent events are tossing two coins, throwing two dice, or drawing a card from each of two different decks of cards. In these examples, the probability of the second event does not change because the first event happened.

Dependent events do affect the outcomes of each other. An example is drawing a card from a deck then drawing another card from the same deck without replacing the first card. The probability of selecting the Queen of Hearts as your first card is $\frac{1}{52}$ (one successful outcome from 52 total cards). But if you don't get the Queen of Hearts as your first card, and the card you do pick is not put back in the deck, the probability of selecting the Queen of Hearts as your second card is $\frac{1}{51}$ (one successful outcome from what are now just 51 cards).

The second difference is that we can find the probability that both Event 1 AND Event 2 will happen, and we can find the probability that either Event 1 OR Event 2 will happen.

To find the probability of Event 1 AND Event 2, we multiply their probabilities. For example, what is the probability of tossing a coin twice and getting heads both times? These are independent events, because the result of the first toss does not change the probability of the second toss. But how likely is it for both tosses to be heads?

Probability of Heads on First Toss AND Second Toss

Probability of Heads on First Toss = $\frac{1}{2}$

Probability of Heads on Second Toss = $\frac{1}{2}$

Probability of Heads on First AND Second Toss = $\frac{1}{2} \text{ X } \frac{1}{2} = \frac{1}{4}$ = 25%

Another way to solve this same problem is to make a chart or table of the possible outcomes. When a coin is tossed twice, there are four possible outcomes:

First Toss	Second Toss
Heads	Heads
Heads	Tails
Tails	Heads
Tails	Tails

We want to find the probability of Heads AND Heads, which we see is one of four possible outcomes, or $\frac{1}{4}$ = 25%.

For another example, suppose we have a bag that contains 6 blue marbles, 3 red marbles and 3 green marbles. If we pick two marbles out of the bag, without replacing them, what are the chances that both marbles will be red? These are dependent events, since not replacing the first marble affects the probability for the second marble. We want to know the chance that both the first AND the second marble will be red, so we will multiply the two probabilities together.

Probability of Choosing Two Red Marbles, without replacement,
from Bag of 6 Blue, 3 Red, and 3 Green

Probability that First Marble is Red = $\frac{3}{12}$

Probability that Second Marble is Red = $\frac{2}{11}$

There are two red marbles left to be selected from a bag of 11 marbles. We assume the first marble was red, or we wouldn't continue picking marbles out of the bag.

Probability that First AND Second Marbles are Red = $\frac{3}{12} \times \frac{2}{11} = \frac{6}{132}$ = 4.5%

To find the probability of Event 1 OR Event 2, we add their probabilities, as long as we eliminate any overlap between them.

For example, if we can draw one card from a deck, replace it, then draw another card from the deck, what is the probability that we will draw either a 4 OR a 7? These are independent events, because the first card is replaced in the deck before choosing the second card. We want to determine whether Event 1 OR Event 2 will happen, so we will add the probabilities together. And finally, there is no overlap in results: none of the fours in the deck are also sevens.

Probability of Drawing a Four OR a Seven from a Card Deck, with replacement

Probability that one draw is a Four = $\frac{4}{52}$

Probability that one draw is a Seven = $\frac{4}{52}$

Probability that EITHER draw will be a Four or a Seven = $\frac{4}{52} + \frac{4}{52} = \frac{8}{52}$ = 15.4%

Consider another example. If we draw one card from a deck, replace it, then draw another card from the deck, what is the probability that we will draw either a 7 or a Heart? These are independent events, because the first card is replaced in the deck before choosing the second card. We want to determine whether Event 1 OR Event 2 will happen, so we will add the probabilities together. But there is some overlap to deduct because one of the Hearts is also a 7. If this overlap is not deducted, we would be double-counting that card and overestimating the probability.

Probability of Drawing a Seven OR a Heart, with replacement

Probability that one draw is a Seven = $\frac{4}{52}$

Probability that one draw is a Heart = $\frac{13}{52}$

Overlap between Hearts and Sevens = $\frac{1}{52}$

Probability of Drawing a 7 OR a Heart = $\frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52}$ = 30.8%

Fundamental Counting Principle

There are many different ways to select and arrange objects. First, here's the fundamental, or basic, counting principle:

If Item A has (a) possible outcomes and Item B has (b) possible outcomes, then there are (ab) possible arrangements, of Item A and Item B. This formula can be extended for an unlimited number of items; just continue to multiply the number of possibilities.

For example, if a restaurant has 5 appetizers, 8 main courses, and 3 desserts, how many different meals could be chosen? Assume that every category is used (skipping dessert is not an option!).

(# Item A Outcomes) x (# Item B Outcomes) x (# Item C Outcomes)

5 appetizers x 8 main courses x 3 desserts =

120 meal possibilities

For another example, suppose you need to select a PIN number for your debit card, using four digits. How many possibilities are there? Each digit can be 0 to 9 (that's 10 choices) and it is possible to use a digit more than once.

4-Digit PIN Numbers, CAN reuse digits

(# Possible 1^{st} Digits) x (# Possible 2^{nd} Digits) x (# Possible 3^{rd} Digits) x (# Possible 4^{th} Digits)

= 10 x 10 x 10 x 10

= 10,000 different PINs

Combinations

Combinations find how many possible outcomes could result when selecting items from a set. In a combination, the order in which the items are selected does not matter.

For example, suppose we go to an ice cream store that has 31 ice cream flavors and we are going to choose two different scoops for our ice cream cone. If we choose vanilla and chocolate, it doesn't matter which is on top (order does not matter), it only matters that we get the combination of those two flavors.

The formula to find the number of combinations is $\frac{n!}{k!(n-k)!}$, where n is the number of items to select from (our 31 flavors) and k is the number of items we will select (our 2 scoops).

Select 2 Scoops from 31 Flavors

# of Combinations = $\frac{n!}{k!(n-k)!}$

$= \frac{31!}{2!(31-2)!}$

$= \frac{31 \, x \, 30 \, x \, \cancel{29!}}{2! \, x \, \cancel{29!}}$

$= \frac{930}{2}$

= 465

Factorial

n! means to multiply the number n by n – 1, by n – 2, by n - 3, and so on down to 1.

So 5! = 5 x 4 x 3 x 2 x 1 (read as "five factorial")

Note that 0! = 1.

In this problem, we don't have to write 31! all the way out, since we notice that it has a common factor of 29! with one of the terms of the denominator. We can simplify the fraction by dividing the numerator and denominator by 29!

So, there are 465 different ways to combine the 31 flavors of ice cream in a two-scoop ice cream cone.

For another example, suppose there are two class times on Saturday at your school, and the same six classes are offered during both times. How many different combinations of classes could you take?

From six classes, select two:

# of Combinations = $\frac{n!}{k!(n-k)!}$

$= \frac{6!}{2!(6-2)!}$

$= \frac{6 \, x \, 5 \, x \, \cancel{4 \, x \, 3 \, x \, 2 \, x \, 1}}{(2 \, x \, 1) \, (\cancel{4 \, x \, 3 \, x \, 2 \, x \, 1})}$

$= \frac{30}{2}$ = 15 combinations

This problem has fewer combinations than the first example, so it won't take as much space to write out the possibilities in a chart and confirm that there are, actually, 15 different combinations of 2 classes.

From six classes, select two:				
Class 1/Class 2				
Class 1/Class 3	Class 2/Class 3			
Class 1/Class 4	Class 2/Class 4	Class 3/Class 4		
Class 1/Class 5	Class 2/Class 5	Class 3/Class 5	Class 4/Class 5	
Class 1/Class 6	Class 2/Class 6	Class 3/Class 6	Class 4/Class 6	Class 5/Class 6
15 different possible combinations				

Permutations

We use permutation formulas when we DO care about the order in which items are selected. The first permutation formula, ${}_nP_n = n!$ finds how many different ways (permutations) n things can be placed in n slots. This formula is for situations where all the items will be in the solution.

For example, suppose there are five people getting in line to give a book report. How many different ways can they line up to be first, second, third, fourth and fifth? All five people will be in the line, so it is not a matter of who will be selected. It is the order that we are concerned with: who will go first, etc.?

How Many Permutations for Five People In Line

${}_nP_n = n!$

${}_5P_5 = 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ permutations

The second permutation formula, ${}_nP_r = \frac{n!}{(n-r)!}$, finds how many different ways n things can be placed in r slots (when r is less than n). The difference from the previous formula is that now we will not select every item in the original set. We are still concerned with the order of the items that are chosen.

For example, suppose 10 people are going to run in a race, and only the fastest 3 people will win a prize for first, second or third. How many different permutations of winners are possible? Only three can win a prize, and it does matter who finishes first, second, or third.

How Many Ways 10 People can Finish in Top 3

${}_nP_r = \frac{n!}{(n-r)!}$

${}_{10}P_3 = \frac{10!}{(10-3)!}$

$= \frac{10 \times 9 \times 8 \times \cancel{7!}}{\cancel{7!}}$

= 720 different permutations of winners

For another example, how many different PIN permutations could be created for a four-digit PIN, using digits 0-9, but where digits can't be repeated. The requirement that digits are not allowed to repeat makes this problem different than the previous example creating a PIN number.

How Many 4-Digit PIN Numbers, if Digits NOT Repeated

${}_nP_r = \frac{n!}{(n-r)!}$ Where n = number of items (the digits 0-9) and

r = number of items to be chosen (4 different digits)

$= \frac{10!}{(10-4)!}$

$= \frac{10 \times 9 \times 8 \times 7 \times \cancel{6!}}{\cancel{6!}}$

= 5,040 different PINs

So if you can't reuse a digit, your possible PIN numbers is reduced from 10,000 possibilities to just 5,040 possibilities. The order selected does matter, though, because the PIN number 1234 is NOT the same permutation as 4321, and only one will help you make a withdrawal from an ATM.

Practice Problems

1. What is the probability of rolling a 6-sided die and getting a 4?

2. If Joe pulls one sock out of a drawer that has 10 blue socks and 6 black socks, what is the probability that he will get a blue sock?

3. Joe's sock drawer contains 10 blue socks and 6 black socks. He pulls two socks out, one at a time, without replacing them. What is the probability that he will get a pair of blue socks?

4. What is the probability of rolling a 6-sided die twice and getting a 4 and a 3?

5. A bag of marbles contains 8 green marbles, 6 red marbles, and 5 yellow marbles. If you draw one marble out, replace it, and then draw another marble out, what is the probability that you get either a red or a green on either try?

6. In a bag of marbles there are 5 blue marbles, 10 red marbles, 3 yellow marbles, and 6 marbles that are a combination of blue and yellow. If you draw one marble out, replace it, and then draw another marble out, what is the probability that: a) the first marble contains blue? b) the second marble contains yellow? c) the first OR second marble contains blue OR yellow?

7. Sally has 5 pairs of pants, 7 shirts, and 2 pairs of shoes. How many different outfits can she wear?

8. A lottery game randomly selects 3 balls with the number 0-9 from three full sets. How many combinations are possible? What is the probability that you will win if you buy one ticket?

9. A salad bar has 15 different items, but you can only choose 4. How many different ways can you make your salad?

10. Bob has 3 trophies for running. How many ways can he order them on his shelf to display them?

11. Bob gets one more trophy, and now he has 4 trophies. Now how many ways can he order them on his shelf to display them?

12. There are 25 students in dance class, and the instructor will randomly select four dancers to go 1st, 2nd, 3rd, and 4th in the recital. How many different ways could they be selected?

Answer Key for Practice Problems

1. $\frac{1}{6}$, or 16.7%	2. $\frac{10}{16}$, or 62.5%	3. $\frac{3}{8}$, or 37.5%	4. $\frac{1}{36}$, or 2.8%	5. $\frac{14}{19}$, or 73.7%
6a. $\frac{11}{24}$, or 45.8%	6b. $\frac{9}{24}$, or 37.5%	6c. $\frac{14}{24}$, or 58.3%	7. 70	8. 1,000; 0.1%
9. 1,365	10. 6	11. 24	12. 303,600	

Practice Problems Solved with Explanation

1. The probability formula is: $\frac{\# \; of \; successful \; outcomes}{total \; \# \; of \; possible \; outcomes}$. A regular die has six sides, numbered from one to six. That means there are 6 total possible outcomes. There is just one that will satisfy this problem—rolling a four. So the probability is $\frac{\# \; of \; successful \; outcomes}{total \; \# \; of \; possible \; outcomes} = \frac{1}{6}$. Convert that fraction to a percent; there is a 16.7% chance of rolling a four.

2. This is also a simple probability problem, using the formula $\frac{\# \; of \; successful \; outcomes}{total \; \# \; of \; possible \; outcomes}$. There are a total of 16 socks in the drawer, so the denominator is 16. Any of the blue socks is considered "success" in this problem, and there are 10 blue socks. That makes the fraction $\frac{10}{16}$, which is 62.5%.

3. This problem involves compound probability because there are two events: picking the first sock and picking the second sock. The events are dependent, because Joe is not going to replace the first sock before he picks the second sock. That means there will be fewer socks in the drawer for the second event. Success for the first event is to pick a blue sock, and there are 10 blue socks out of a total of 16 socks in the drawer. So the probability for the first event is $\frac{10}{16}$. Assuming that there was success on the first pick, success for the second pick is to get one of the 9 remaining blue socks out of a total of 15 remaining socks. The probability for the second event is $\frac{9}{15}$. We want to know the probability that the first event AND the second event will happen, so we multiply the two probabilities. $\frac{10}{16} \times \frac{9}{15} = \frac{90}{240}$. The fraction simplifies to $\frac{3}{8}$, or 37.5%

4. This problem is also compound because there are two rolls of the die. The events are independent, because the outcome of the second roll is not affected by the first roll. The probability of rolling a 4 is $\frac{1}{6}$, because four is one successful outcome out of six total possible outcomes. The probability of rolling a 3 is also $\frac{1}{6}$. To find the probability that both events will happen, multiply the probabilities. $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$. Converting to a percent gives us 2.8%.

5. This problem is also compound probability, with two events. Since the first marble will be replaced, the events are independent (they do not affect each other). There are a total of 19 marbles in the bag, and the first success is drawing one of the 6 red marbles. That probability is $\frac{6}{19}$. There will still be 19 marbles in the bag for the second draw, and this time success is getting one of the 8 green marbles. Probability for this event is $\frac{8}{19}$. We want to find the probability that EITHER Event 1 OR Event 2 will happen, so we add the probabilities. $\frac{6}{19} + \frac{8}{19} = \frac{14}{19}$. Remember with EITHER/OR problems we need to determine whether there is any overlap that needs to be removed. In this example there is no overlap, because none of the red marbles are also green marbles. So the probability is $\frac{14}{19}$, or 73.7%.

6. The events are independent, because the first marble will be replaced. The probability of the first event (drawing any marble with blue) is $\frac{11}{24}$. That is because there are 24 total marbles, and both the 5 all-blue marbles and the 6 blue-yellow marbles are success, and 5 + 6 = 11. Success for the second draw, which is getting any marble with yellow, is $\frac{3+6}{24} = \frac{9}{26}$. We can't just add these two together to get the probability of either marble containing blue OR yellow because we have to eliminate the overlap of the 6 blue-yellow marbles, or $\frac{6}{24}$. That probability is $\frac{11}{24} + \frac{9}{24} - \frac{6}{24} = \frac{14}{24}$, which is 58.3%.

7. Sally's outfit problem uses the formula: (# A Outcomes) x (# B Outcomes) x (# C Outcomes) = Total Combinations. The first outcome is 5 different pairs of pants, which we multiply by 7 different shirts, and then multiply by 2 pairs of shoes. 5 x 7 x 2 = 70. That means Sally can combine her clothes in 70 different ways.

8. The lottery problem uses the same formula as the previous question. There are 10 outcomes for the first ball; 10 outcomes for the second ball; and 10 outcomes for the third ball. That makes 10 x 10 x 10 = 1,000 different outcomes. If you buy one lottery ticket, you have one successful outcome out of 1,000. The probability of winning is $\frac{1}{1000}$, which equals 0.1%.

9. The salad bar problem is a combination. It does not matter in what order we choose our toppings, but we can only choose 4 out of the 15 total possibilities. The formula for combinations is $\frac{n!}{k!(n-k)!}$, where the variable "n" is the total number of possibilities, and the variable "k" is the quantity we will choose. Remember that n! is read "n factorial" and it means to multiply the number n by n – 1, then n – 2, then n – 3, all the way down to 1.

So, $\frac{n!}{k!(n-k)!} = \frac{15!}{4!(15-4)!} = \frac{15 \, x \, 14 \, x \, 13 \, x \, 12 \, x \, \cancel{11}!}{4 \, x \, 3 \, x \, 2 \, x \, 1 \, x \, \cancel{11}!} = \frac{32{,}760}{24}$ = 1,365. Notice that we didn't have to write 15! all the way out because we see that the denominator will have a factor of 11!. There are 1,365 different ways to choose four items from the 15-item salad bar.

10. Bob's trophies are a permutation problem because we do care what order he places them in. Since he will display all his trophies, we use the formula $_nP_n$ = n!. He has 3 trophies, so n = 3. n! = 3 x 2 x 1 = 6. There are six ways he can arrange the three trophies on his shelf.

11. This is really the same problem as the last one, but now n = 4. We still use the formula $_nP_n$ = n!. 4! = 4 x 3 x 2 x 1 = 24. With just one more trophy, the situation changes from 6 different permutations to 24 different possible permutations.

12. This problem is also a permutation since it matters what order the dancers will be selected (1st, 2nd, 3rd, and 4th). And this time we are only choosing 4 of the 25 dancers, so we will use the formula $_nP_r = \frac{n!}{(n-r)!}$. The variable n is the total number of items we are choosing from (the 25 dancers). The variable r is the number we will choose (4 dancers). We will be able to factor out 21! from both numerator and denominator.

So $_nP_r \frac{n!}{(n-r)!} = \frac{25!}{(25-4)!} = \frac{25 \, x \, 24 \, x \, 23 \, x \, 22 \, x \, \cancel{21}!}{\cancel{21!}}$ = 303,600.