A Review of Factors, Like-terms & Distributive Property

This section covers more advanced factoring, so first review three concepts covered in earlier section: factors, like-terms and the Distributive Property.

"Factors" are the smaller numbers that can be multiplied together to get a larger number. (See Section 1 to review.) The factors of 12 are 1, 2, 3, 4, 6 and 12. The number 12 can be divided evenly by each of these numbers. If something contains one or more variables, those are also factors. For example, the factors of $(12x^2y)$ are 1, 2, 3, 4, 6, 12, x and y.

Section 6 covers "like-terms." Remember that "terms" are the pieces of an expression that are separated by addition and subtraction. "Like-terms" have the same variables with the same exponents. So $4x^2$ and $9x^2$ are like-terms because they both contain the variable "x" with the same exponent "2." However, $4x$ and $9x^2$ are not like-terms because the exponent on the variable is not the same.

The Distributive Property (see Section 1) tells us what to do when multiplying or dividing two or more terms by another number. Consider (b + c). The Distributive Property says that if we multiply that expression by another value, a, we will get ab + ac. The Distributive Property also says that if we divide that expression by the value a, we get $\frac{b}{a} + \frac{c}{a}$.

<u>Distributive Property</u>

Multiplication: $a(b + c) = ab + ac$

Division: $\frac{b+c}{a} = \frac{b}{a} + \frac{c}{a}$

More Advanced Examples of Combining of Like-terms

To really understand the concept of combining like-terms, practice with some examples.

Combine the like-terms in this expression:

$\underline{5} + 2x - 3y + x^2 - 3x^2 + 4xy + \underline{3}$	Starting with the numbers without variables, 5 + 3 = 8.
$8 + 2x - 3y + \underline{x^2} - \underline{3x^2} + 4xy$	Next combine the terms with x^2. Remember x^2 by itself is the same as $1x^2$, and $1x^2 - 3x^2 = -2x^2$. (Review Section 5 if necessary.)
$8 + 2x - 3y - 2x^2 + 4xy$	
$-2x^2 + 2x + 4xy - 3y + 8$	Rearrange the terms to make the expression easier to read.

Underlining the like-terms, as we did in this example, makes it easier to see which terms to combine. It does not matter what terms you start with. You can work left to right, or you can start with terms with the highest exponents. For your final answer, the common practice is to rearrange the terms so the variables are alphabetized and terms with higher exponents are listed first.

Combine the like-terms in this expression:

$\underline{-2x^2} + 3x \underline{- x^2} + 4x + 5 + 7x$ Starting with the terms with the highest exponent: $-2x^2 - x^2 = -3x^2$.

$-3x^2 + \underline{3x} + \underline{4x} + 5 + \underline{7x}$ Next combine the terms with x. $3x + 4x + 7x = 14x$.

$-3x^2 + 14x + 5$

Combine the like-terms in this expression:

$3x^2 + \underline{4y^4} + 7x + y^3 \underline{- 2y^4}$ There is just one set of like-terms. $4y^4 - 2y^4 = 2y^4$.

$3x^2 + 2y^4 + 7x + y^3$

$3x^2 + 7x + 2y^4 + y^3$ We can rearrange the terms for readability though.

More Advanced Examples of Distributive Property

We can distribute a value over more than two terms; in fact, we can distribute a value over any number of terms. We just do each one at a time, being careful with rules for positive and negative operations.

Look at $4(2x - y + 7)$. To "distribute" the 4, multiply 4 by each term one at a time.

First multiply $4(2x) = 8x$
then $4(-y) = -4y$
and then $4(7) = 28$.
Put it all together, and the answer is $8x - 4y + 28$.

And look at $-9x(2x + 3y - 5)$. Multiply each term by -9x one at a time.

$-9x(2x) = -18x^2$.
$-9x(3y) = -27xy$
$-9x(-5) = 45x$

Put these together, and we get $-18x^2 + 45x - 27xy$.

At first, this example may not look like we need the distributive property, but we do!

$(5x + 7) - (3x + 2)$ There's a subtraction sign before the second parentheses. Before we remove the parentheses, distribute -1 across the terms in the those parentheses.

$(5x + 7) + -1(3x + 2)$

$\underline{5x} + 7 + \underline{-3x} - 2$ Combine the x terms; $5x - 3x = 2x$.

$2x + \underline{7} - \underline{2}$ Combine the numeric terms; $7 - 2 = 5$.

$2x + 5$

Isn't there an easier way to do that? Well, yes, there is a shortcut. But distribution is why it works!

When subtracting terms in parentheses,
change the sign of each term:

(3x + 4) – (2x – 7)

3x + 4 - 2x + 7

X + 11

Factoring Out the Greatest Common Factor

Now that we know how to put things together, it's easier to take them apart. In math, we usually try to "simplify" things by getting them in their most basic forms. With fractions, we simplify by finding common factors. We do the same to simplify expressions. Factoring out the greatest common factor is almost always the first step to consider, even for the most complicated expressions.

To simplify an expression, we look for the greatest common factor shared by the terms. Then we use the distributive property (doing the reverse of the previous examples).

Simplify (2x – 8) The factors of 2x are 2 and x.
The factors of 8 are 1, 2, 4, and 8.
The greatest factor that the terms have in common is 2, so divide both
2(x-4) terms by the common factor, which "moves" outside the parentheses.
If we distribute that again, we are back to (2x – 8).

Notice we really didn't have to consider that the second term was negative, or that there can be negative factors. We start with the positive factors.

Simplify $3x^2 + 12xy – 6x$ First, notice that each term has a numeric coefficient divisible by 3.
In addition, there is an x variable in every term. So, factor out 3x.

$$\frac{3x2}{3x} + \frac{12xy}{3x} - \frac{6x}{3x}$$

3x(x + 4y – 2) Each term is divided by 3x, which moves outside the parentheses.

Multiplying with FOIL (First, Outer, Inner, Last)

The acronym FOIL stands for "First, Outer, Inner, Last," and it helps us remember how to distribute two groups that each with two terms. For example, consider how we would distribute $(x + 1)(x - 2)$. Each term in the first parentheses must be multiplied by each term in the second parentheses.

$(x + 1)(x - 2)$	First: $x(x) = x^2$
$(x + 1)(x - 2)$	Outer: $x(-2) = -2x$
$(x + 1)(x - 2)$	Inner: $1(x) = x$
$(x + 1)(x - 2)$	Last: $1(-2) = -2$
$x^2 - 2x + x - 2$	Result
$x^2 - x - 2$	Combine Like-terms

In the example above, the terms are multiplied together in the FOIL sequence. The FIRST terms in each group are multiplied together: $x(x) = x^2$. Then the OUTER terms $x(-2) = -2x$. Don't forget to follow the rules of multiplying positive and negative numbers. INNER terms are $1(x) = x$. Then the LAST terms are $1(-2) = -2$. Combining all those results gives us an expression with some like-terms. When those like-terms are combined, our final result is $x^2 - x - 2$.

Here is another example of multiplying using the FOIL sequence.

$(2x - 3)(x - 2)$ First: $2x(x) = 2x^2$; Outer: $2x(-2) = -4x$; Inner: $-3(x) = -3x$; Last: $-3(-2) = 6$

$2x^2 \underline{- 4x} \underline{- 3x} + 6$ Combine like-terms.

$2x^2 - 7x + 6$

Factoring with FOIL when x^2 Coefficient = 1

In the previous examples, we multiplied groups with two terms ("binomials") and got answers with three terms ("trinomials"). Next we will do the opposite: start with a trinomial and break it down (factor it) into two binomials.

This first method works when the "coefficient" (the number multiplied by) the x^2 term is 1. Remember, when no number is written next to a variable, it's understood that it is a 1. Factor $x^2 - x - 2$.

Look at the Coefficients of the Trinomial

$x^2 - x - 2$

Result of FIRST: 1

Result of INNER & OUTER together: -1

Result of LAST: -2

Factor $x^2 - x - 2$

$(x \quad)(x \quad)$ — Start by drawing the parentheses and put an "x" as the FIRST part of each term. This will be the FIRST terms that multiply together for a result of x^2.

Next, look at the result of multiplying the LAST numbers: -2.
We need two numbers (factors) that <u>multiply</u> together for a result of -2.
The factors of 2 are: 1, 2.
To get a negative result, one has to be negative and one has to be positive.

Which of those options add OUTER and INNER together to result in -1?

-1 + 2 = 1: No, not the result we are looking for.

$(x + 1)(x - 2)$ — 1 + -2 = -1: Yes! So we fill the parentheses with +1 and -2

> To summarize:
>
> Look for factors of the numeric-only term.
>
> Which factors can be MULTIPLIED to get that number and ADDED to get the coefficient of the x term?

Factor $x^2 - 10x + 24$

$(x \quad)(x \quad)$ — Write the parentheses and the FIRST terms.

Factors of 24: (1)(24); (2)(12); (3)(8); (4)(6).
To get positive 24, either both must be positive or both must be negative.
Which combination will add together to get -10? Try -4 and -6.

$(x - 4)(x - 6)$ — Check with FOIL: $x^2 - 6x - 4x + 24$, which equals $x^2 - 10x + 24$.

Factor $x^2 + 2x - 8$

(x) (x)	Factors of 8: (1)(8); (2)(4). To get negative eight, one must be positive and one must be negative. Which combination adds together to get 2? Try -2 and +4.
(x – 2) (x + 4)	Check with FOIL: $x^2 + 4x - 2x - 8$, which equals $x^2 + 2x - 8$.

Factoring with FOIL when x^2 Coefficient ≠ 1

Factoring is nearly the same when the x^2 coefficient is not one. The difference is that the x^2 coefficient must be combined with the factors for the FIRST and OUTER terms.

For example, factor: $3x^2 - 13x - 10$

(3x) (x)	Still begin with parentheses and the FIRST terms. The only factors of 3 are 1 and 3, so one of the first terms is 3x and one is x: $(3x)(x) = 3x^2$. Factors of 10: (1)(10); (2)(5). To get negative 10, one must be positive and one must be negative. Also, one of them will be multiplied by 3 before adding to get -13. Try (+2) and (3)(-5).
(3x + 2) (x -5)	Check with FOIL: $3x^2 - 15x + 2x - 10$, which equals $3x^2 - 13x - 10$.

Factor: $6x^2 - 7x - 20$

(__x) (__x)	This time the FIRST terms could be (6x)(x) or (3x)(2x). We will try (6x)(x) first, as a trial-and-error attempt.
(6x) (x)	Factors of 20: (1)(20); (2)(10); (4)(5). One must be positive; one negative. And, one must be multiplied by 6 before combining to get -7. (6)(1) ± (20). (1) ±(6)(20). (6)(2)±(10). (2)±6(10). (6)(4)±(5). (4)±(6)(5).
(3x) (2x)	None of those options had a total or difference of 7, so instead try (3x)(2x). (3)(1)±(2)(20). (2)(1)±(3)(20). (3)(2)±(2)(10). (2)(2)±(3)(10). (3)(4)±(2)(5). (2)(4)±(3)(5). Aha! 8 – 15 = -7, we are close!
(3x + 4) (2x – 5)	We want our INNER and OUTER to result in +8 and -15. Check with FOIL: $6x^2 - 15x + 8x - 20$, which equals $6x^2 - 7x - 20$.

There is some guesswork involved in solving these problems. With some experience, your guesses will become more educated and less random. Just keep checking each option with FOIL and you will get the correct answer.

Difference Between Squares

There is a special case of factoring with FOIL that starts with a binomial (just two terms) instead of a trinomial (three terms). This example happens when each term is a number that is a perfect square.

For example, factor: $x^2 - 4$

(x) (x) Still begin with parentheses and the FIRST terms.

The factors of 4 are (1)(4); (2)(2).
To get negative 4, one must be positive and one must be negative.
(x + 2) (x – 2) And, really, the "middle term" in $x^2 - 4$ is 0x; it has cancelled out with +2 and -2.

When we check with FOIL: $x^2 - 2x + 2x - 4$, which equals $x^2 - 4$.

For another example, factor $9x^2 - 16$.

(3x – 4) (3x + 4) As soon as you recognize that everything in both terms is a perfect square, you can immediately use the square roots for your terms.

Check with FOIL: $9x^2 + 12x - 12x - 16$, which equals $9x^2 - 16$.

Practice Problems

1. Simplify by combining like-terms: $2x + 4y - 7x + 2 + 4y$
2. Simplify by combining like-terms: $x^2 - 4xy + 3x^2 + 4y - x^2 + x^3$
3. Distribute: $3x(x + 2)$
4. Simplify by combining like-terms: $(x^2 + x) - (3x^2 + 7)$
5. Factor: $16x^2 - 8x$
6. Factor: $2x^3 - 12x^2 + 4y - 2$
7. Distribute: $(x - 4)(x + 3)$
8. Factor: $x^2 - x - 20$
9. Factor: $3x^2 + 12x - 36$
10. Factor: $8x^2 + 6x - 2$
11. Factor: $x^2 - 25$
12. Factor: $9x^2 - 4$

Answer Key for Practice Problems

1. $-5x + 8y + 2$	2. $x^3 + 3x^2 - 4xy + 4y$	3. $3x^2 + 6x$
4. $-2x^2 + x - 7$	5. $(8x)(2x - 1)$	6. $2(x^3 - 6x^2 + 2y - 1)$
7. $x^2 - x - 12$	8. $(x - 5)(x + 4)$	9. $3(x - 2)(x + 6)$
10. $2(x + 1)(4x - 1)$	11. $(x - 5)(x + 5)$	12. $(3x - 2)(3x + 2)$

Practice Problems Solved with Explanation

1. $\underline{2x} + 4y \underline{- 7x} + 2 + 4y$	Start with the x terms: $2x - 7x = -5x$
$-5x \underline{+ 4y} + 2 \underline{+ 4y}$	Combine the y terms: $4y + 4y = 8y$
$-5x + 8y + 2$	
2. $\underline{x^2} - 4xy + \underline{3x^2} + 4y \underline{- x^2} + x^3$	Move the x^3 term to the beginning, and combine the x^2 terms.
$x^3 + 3x^2 - 4xy + 4y$	None of the other terms can be combined.
3. $3x(x + 2)$	Multiply $3x(x) = 3x^2$; multiply $3x(2) = 6x$.
$3x^2 + 6x$	
4. $(x^2 + x) - (3x^2 + 7)$	Distribute the implied -1 first, which is the same as changing the sign of each term in the second group of parentheses.
$\underline{x^2} + x \underline{- 3x^2} - 7$	Combine the x^2 terms.
$-2x^2 + x - 7$	
5. $16x^2 - 8x$	Both terms contain an x; both have a factor of 8, so factor out 8x.
$8x(2x - 1)$	
6. $2x^3 - 12x^2 + 4y - 2$	The only thing <u>every</u> term contains is a factor of 2, so factor out 2.
$2(x^3 - 6x^2 + 2y - 1)$	
7. $(x - 4)(x + 3)$	Following FOIL, the FIRST terms are $(x)(x) = x^2$. The OUTER terms are $(x)(3) = 3x$. The INNER terms are $(-4)(x) = -4x$, The LAST terms are $(-4)(3) = -12$.
$x^2 + 3x - 4x - 12$	Combine like-terms.
$x^2 - x - 12$	

8. $x^2 - x - 20$

(x) (x) — Start with the parentheses and the x terms.
Factors of 20 are: (1)(20); (2)(10); (4)(5).
To get negative 20, one must be positive and one must be negative.
The combination that adds together to get -1 is -5 and +4.

$(x - 5)(x + 4)$ — Check with FOIL: $x^2 + 4x - 5x - 20$, which equals $x^2 - x - 20$

9. $3x^2 + 12x - 36$ — Each term includes a factor of 3, so remove that first.

$3(x^2 + 4x - 12)$

3(x) (x) — Factors of 12 are: (1)(12); (2)(6); (3)(4).
To get -12, one factor must be positive and one negative.
The combination that adds together to get +4 is -2 and +6.

$3(x - 2)(x + 6)$ — The 3 remains outside the other parentheses.

Check with FOIL: $3(x^2 + 6x - 2x - 12)$
$3(x^2 + 4x - 12)$
$3x^2 + 12x - 36$

10. $8x^2 + 6x - 2$ — Each term includes a factor of 2, so remove that first.

$2(4x^2 + 3x - 1)$

2(__x) (__x) — The coefficient of x^2 term is 4, which might factor as (1)(4) or (2)(2).
The last term is 1, though, so an educated guess to get the other terms right is to use (1)(4) for the x-term coefficient.

2(4x) (x) — The factors of 1 are just (1)(1), with one positive and one negative.

$2(4x - 1)(x + 1)$ — Check with FOIL: $2(4x^2 + 4x - x - 1)$
$2(4x^2 + 3x - 1)$
$8x^2 + 6x - 2$

11. $x^2 - 25$ — If you recognize immediately that x^2 and 25 are perfect squares, try the square roots as the factors.

$(x - 5)(x + 5)$ — Check with FOIL: $x^2 + 5x - 5x - 25$, which equals $x^2 - 25$.

12. $9x^2 - 4$ — Another example with perfect squares.

$(3x - 2)(3x + 2)$ — Use the square root of 9, which is 3, as the x coefficients.